

Bosonization in higher dimensions via noncommutative field theory

Alexios P. Polychronakos

Physics Department, City College of the CUNY, New York, NY 10031
alexios@sci.ccny.cuny.edu

We propose the bosonization of a many-body fermion theory in D spatial dimensions through a noncommutative field theory on a $(2D - 1)$ -dimensional space. This theory leads to a chiral current algebra over the noncommutative space and reproduces the correct perturbative Hilbert space and excitation energies for the fermions. The validity of the method is demonstrated by bosonizing a two-dimensional gas of fermions in a harmonic trap.

PACS-98: 04.70.Dy, 11.15.-q

Bosonization of one-dimensional fermion systems is a well-established technique in condensed matter and field theory [1,2]. Its usefulness lies in the fact that the low-energy dynamics of the equivalent bosonic theory encodes collective excitations of the fermion system and thus provides a handle to analyzing strongly correlated fermions.

The success of the method relies on special properties of one-dimensional spaces. Intuitively, the fermion dynamics around the Fermi (or Dirac) sea become tantamount to one-dimensional wave propagation, the corresponding phonon states encoding quantum excitations [3]. This leads to a mapping of states between the two systems both at the many-body and the field theory (operator) level.

There are several approaches to extending these techniques to higher dimensions [4,5]. The most obvious way to proceed is by generalizing the Fermi sea idea ‘point-wise’ in the extra dimensions, giving rise to the so-called radial bosonization. This approach, however, has some obvious drawbacks that are at the heart of the difficulties with higher dimensions. Specifically, it produces one tower of states for each point in the extra dimensions, thus overcounting the degrees of freedom by not taking into account the ‘finite cell’ structure of the sea. Improvements to the method have been proposed, but bosonization of higher dimensional fermion systems remains to a large extent an open issue.

The purpose of this Letter is to propose a theory that addresses the above problem and provides a potentially exact bosonization scheme for higher dimensional theories. The approach is through a phase space description of the many-body fermion problem. For D spatial dimensions this leads to a $2D - 1$ -dimensional nonlocal bosonic theory. (In this respect, our approach is closest to the one in [5].) Only in one dimension the dimensionalities agree and the theory becomes local. We accept this as the price of the (nontrivial) payout of higher-dimensional

bosonization.

We shall present here an outline of the new method, motivated by the semiclassical phase space droplet picture, and demonstrate it in a simple, solvable situation. This would make its physical meaning and content as accessible as possible. A full derivation will be given in a subsequent publication.

We shall consider a collection of spinless, noninteracting fermions in $D = d + 1$ spatial dimensions. (Spin and interactions can be added later without conceptual problems.) The starting point will be the semiclassical phase space droplet description of the system [6], in the hamiltonian formulation derived and fully exposed in [7]. We review here the basic notions and quote the relevant results.

Particles move on a phase space of dimension $2d + 2$ with coordinates ϕ^μ and a single-particle Poisson structure

$$\{\phi^\mu, \phi^\nu\}_{sp} = \theta^{\mu\nu}, \quad \mu, \nu = 1, \dots, 2d + 2 \quad (1)$$

For simplicity we shall assume that we have chosen Darboux coordinates and momenta $\phi^\mu = \{x^i, p^i\}$ so that $\theta^{\mu\nu}$ is in the canonical form, with determinant $\det \theta^{\mu\nu} = 1$. Particle dynamics are fixed by a single-particle hamiltonian $H_{sp}(\phi^\mu)$.

The system can be described semiclassically by a state in which each cell of volume $(2\pi\hbar)^{d+1}$ in phase space is filled by a single fermion. This leads to a phase space ‘droplet’ picture of constant density $\rho_o = 1/(2\pi\hbar)^{d+1}$, determined by its $2d + 1$ -dimensional boundary. We choose to parametrize the boundary in terms of one of the phase space coordinates, say p^1 , which becomes a function of the remaining variables $R(\phi^\mu)$ ($\phi^\mu \neq p^1$). It is convenient to separate in notation the variable conjugate to p^1 , $x^1 \equiv \sigma$ ($\{p^1, x^1\}_{sp} = -1$), and use middle Greek indices for the remaining $2d$ phase space variables ϕ^α and corresponding Poisson structure $\theta^{\alpha\beta}$. ϕ will collectively denote the coordinates ϕ^α .

The dynamics of the system are given in terms of the canonical Poisson brackets of the field $R(\sigma, \phi)$ (not to be confused with the single-particle brackets $\{.,.\}_{sp}$). Using the shorthand $R_i = R(\sigma_i, \phi_i)$, these are derived as [7]:

$$\{R_1, R_2\} = \frac{1}{\rho_o} [-\delta'(\sigma_-)\delta(\phi_-) - \delta(\sigma_-)\theta^{\alpha\beta}\partial_\alpha R_+\partial_\beta\delta(\phi_-)] \quad (2)$$

with (σ_-, ϕ_-) and (σ_+, ϕ_+) relative and mid-point coordinates respectively. The hamiltonian for the field R is the sum of the single-particle hamiltonians over the bulk of the droplet

$$H = \rho_o \int dp^1 d\sigma d^{2d}\phi H_{sp}(\sigma, \phi) \vartheta(R - p^1) \quad (3)$$

where $\vartheta(x) = \frac{1}{2}[1 + \text{sgn}(x)]$ is the step function. (2) and (3) define a bosonic field theory (in a hamiltonian setting) that describes the system semiclassically.

The above theory already represents an improvement over previous methods, which essentially amounted to keeping only the first (constant) term in the Poisson brackets. Although at this stage it still suffers from shortcomings similar to the ones of previous Fermi sea approaches, it will serve as the starting point for an exact bosonization. Below we shall simply motivate the form of the exact theory and demonstrate its merits, leaving a full derivation and details for a future publication.

We start by noticing that R_+ in the right hand side of the above Poisson brackets can also be put to R_1 (or R_2), since the antisymmetry of $\theta^{\alpha\beta}$ makes the difference irrelevant. In view of this, the right hand side can be recast using the single-particle Poisson brackets for the variables ϕ_1^α as

$$\theta^{\alpha\beta}\partial_\alpha R_+\partial_\beta\delta(\phi_-) \rightarrow \{R(\sigma_1, \phi_1), \delta(\phi_1 - \phi_2)\}_{sp1} \quad (4)$$

We now make the central claim of this paper: the exact theory will be obtained from the above expression by ‘quantizing’ (at the classical level) the remaining phase space coordinates; that is, by promoting the single-particle Poisson brackets to noncommutative Moyal brackets $\{.,.\}_*$ on the $2d$ -dimensional phase space manifold ϕ^α :

$$\{R_1, R_2\} = \frac{1}{\rho_o} [-\delta'(\sigma_-)\delta(\phi_-) - \delta(\sigma_-)\{R_1, \delta(\phi_-)\}_{*1}] \quad (5)$$

The Moyal brackets between two functions of ϕ are expressed in terms of the noncommutative Groenewald-Moyal star-product on the phase space ϕ [8]:

$$\{F(\phi), G(\phi)\}_* = \frac{1}{i\hbar} [F(\phi) * G(\phi) - G(\phi) * F(\phi)] \quad (6)$$

with \hbar itself being the noncommutativity parameter. Correspondingly, the hamiltonian H is given by expression (3) but with star-products replacing the usual products between its terms.

Inherent in the definition of the star product is a one-to-one mapping between classical functions $F(\phi)$ and operators \hat{F} on a first-quantized phase space. Specifically, define the set of quantum operators $\hat{\phi}$ corresponding to the residual classical phase space, satisfying

$$[\hat{\phi}^\alpha, \hat{\phi}^\beta] = i\hbar\theta^{\alpha\beta}, \quad \alpha, \beta = 1, \dots, 2d \quad (7)$$

and choose a basis $|a\rangle$ for the (single-particle) quantum mechanical Hilbert space on which the above operators act. This could be, e.g., the Fock space of d harmonic oscillators, or any other basis. Classical commutative functions map to operators by adopting a specific operator ordering : $F(\hat{\phi})$: in the expression for $F(\hat{\phi})$, and star product maps to operator product:

$$F(\phi) \leftrightarrow \hat{F} = : F(\hat{\phi}) : , \quad (F * G)(\phi) \leftrightarrow : \hat{F}\hat{G} : \quad (8)$$

It is customary to choose the fully symmetric Weyl ordering

$$: e^{iq^\beta \hat{\phi}^\beta} := e^{iq^\beta \hat{\phi}^\beta}, \quad q \in R^{2d} \quad (9)$$

in which case the corresponding star product is given in terms of the Fourier transform of the functions as

$$(\tilde{F} * \tilde{G})(k) = \int \frac{d^{2d}q}{(2\pi)^{2d}} \tilde{F}(k - q) \tilde{G}(q) e^{\frac{i}{2}\hbar\theta^{\alpha\beta}q_\alpha k_\beta} \quad (10)$$

Any other appropriate ordering can be used, leading to alternative definitions of the star product.

In the limit $\hbar \rightarrow 0$ the above becomes the usual (commutative) convolution integral in Fourier space, while the brackets (6) go over to standard Poisson brackets on the manifold. For nonzero \hbar the star-product and corresponding Moyal brackets are nonlocal. The obtained theory is, therefore, nonlocal in the $2d$ dimensional phase space while it remains local in the σ direction.

To analyze the content of the above theory, it is beneficial to perform a change of variable in the field R to a matrix representation. We define the mapping between functions $F(\phi)$ and matrix elements $F^{ab} = (a|\hat{F}|b)$:

$$F^{ab} = \int d^{2d}\phi F(\phi) C_{ab}(\phi), \quad F(\phi) = \sum_{a,b} F^{ab} C_{ba}(\phi) \quad (11)$$

with $C_{ab}(\phi)$ an appropriate set of basis functions depending on the ordering:

$$C_{ab}(\phi) = \int \frac{d^{2d}q}{(2\pi)^{2d}} e^{iq^\alpha \phi^\alpha} (a| : e^{-iq^\beta \hat{\phi}^\beta} : |b) \quad (12)$$

In this representation, star products become matrix multiplication and phase space integrals become traces:

$$(F * G)^{ab} = F^{ac} G^{cb}, \quad \int d^{2d}\phi F(\phi) = (2\pi\hbar)^d F^{aa} \quad (13)$$

with repeated indices summed.

The above mapping can be applied to the classical dynamical field $R(\sigma, \phi)$ mapping it to dynamical matrix elements $R^{ab}(\sigma)$. To express the full Poisson brackets for R in this representation we need the matrix expression for the delta function $\delta(\phi_1 - \phi_2)$, with defining property

$$\int d^{2d}\phi_1 F(\phi_1) \delta(\phi_1 - \phi_2) = F(\phi_2) \quad (14)$$

Since $\delta(\phi_1 - \phi_2)$ is a function of two variables, its matrix transform in each of them will produce a symbol with four indices $\delta^{a_1 b_1; a_2 b_2}$. The above defining relation in the matrix representation becomes

$$(2\pi\hbar)^d F^{a_1 b_1} \delta^{b_1 a_1; a_2 b_2} = F^{a_2 b_2} \quad (15)$$

which implies

$$\delta^{a_1 b_1; a_2 b_2} = \frac{1}{(2\pi\hbar)^d} \delta^{a_1 b_2} \delta^{a_2 b_1} \quad (16)$$

With the above, and using $\rho_o = 1/(2\pi\hbar)^{d+1}$, the canonical Poisson brackets of the matrix R^{ab} become

$$\{R_1^{ab}, R_2^{cd}\} = -2\pi\hbar\delta'(\sigma_-)\delta^{ad}\delta^{cb} + 2\pi i\delta(\sigma_-)(R_1^{ad}\delta^{cb} - R_1^{cb}\delta^{ad}) \quad (17)$$

Upon quantization of the theory, the fields $R^{ab}(\sigma)$ become operators whose quantum commutator is given by the above Poisson brackets times $i\hbar$. Defining, further, the Fourier modes

$$R_k^{ab} = \int \frac{d\sigma}{2\pi\hbar} R^{ab}(\sigma) e^{-ik\sigma} \quad (18)$$

the quantum commutators become

$$[R_k^{ab}, R_{k'}^{cd}] = k\delta(k+k')\delta^{ad}\delta^{cb} - R_{k+k'}^{ad}\delta^{cb} + R_{k+k'}^{cb}\delta^{ad} \quad (19)$$

The Casimir $R_0^{aa} \equiv N$ is the total fermion number. For a compact dimension σ , normalized to a circle of length 2π , the Fourier modes become discrete.

This is nothing but a chiral current algebra for the matrix field R_k^{ab} on the unitary group of transformations

of the first-quantized states $|a\rangle$. To make this explicit, consider for the moment that the first-quantized Hilbert space is K -dimensional, that is, $a, b, c, d = 1, \dots, K$ (this would be the case for a compact phase space $\{\phi^\alpha\}$), and define the hermitian $K \times K$ fundamental generators of $U(K)$, T^A , $A = 0, \dots, K^2 - 1$, normalized as $\text{tr}(T^A T^B) = \frac{1}{2}\delta^{AB}$ and defining the $U(M)$ structure constants $[T^A, T^B] = if^{ABC}T^C$ with $f^{0AB} = 0$. Using the T^A as a basis we express the quantum commutators (19) in terms of the $R^A = \text{tr}(T^A R)$ as

$$[R_k^A, R_{k'}^B] = \frac{1}{2}k\delta(k+k')\delta^{AB} + if^{ABC}R_{k+k'}^C \quad (20)$$

This is the so-called Kac-Moody algebra for the group $U(K)$.

The coefficient k_{KM} of the central extension of the Kac-Moody algebra (the first, affine term) must be quantized to an integer to have unitary representations. Interestingly, this coefficient in the above commutators emerges quantized to the value $k_{KM} = 1$. This is crucial for bosonization [2]. The $k_{KM} = 1$ algebra has a unique irreducible unitary representation over each ‘vacuum’; that is, over highest weight states annihilated by all $R^A(k)$ for $k > 0$ and transforming under a fully antisymmetric $SU(K)$ representations under $T^A(0)$. We argue that *these Fock-like representations correspond exactly to the perturbative Hilbert space of excitations of the many-body fermionic system over the full set of possible Fermi sea ground states*. The $U(1)$ charge R_0^0 , which is a Casimir, corresponds to the total fermion number; diagonal operators R_k^H , for $k < 0$ and H in the Cartan subgroup of $U(K)$ generate ‘radial’ excitations in the Fermi sea; while off-diagonal operators R_k^T , for $k < 0$ and T off the Cartan subgroup, generate transitions of fermions between different points of the Fermi sea.

Finally, the hamiltonian of the bosonic theory becomes

$$H = \int \frac{dp^1 d\sigma}{2\pi\hbar} \text{tr} H_{sp}(\sigma, p^1, \hat{\phi}) \vartheta(R - p^1) \quad (21)$$

where p^1 remains a scalar integration parameter while $\hat{\phi}$ and R are matrices as before. Clearly there are issues of ordering in the above expression, both quantum and matrix, just as in standard 1 + 1-dimensional bosonization.

It is clear that for $D = 1$ our theory reproduces standard bosonization. In this case there is no residual phase space and we recognize $R(\sigma)$ as the derivative of the bosonized field $\partial_\sigma \Phi$ representing the chiral density in one dimension. (Two independent chiral fields R_\pm , representing the two intercepts of the compact droplet boundary with the line $x=\text{constant}$, are needed.) The expression

of the hamiltonian (21) for $H_{sp} = p$ or $H_{sp} = p^2/2$ assumes the usual relativistic or nonrelativistic bosonized form $H = (\partial_x \Phi)^2/2$ or $H = (\partial_x \Phi)^3/6$, respectively, fully recovering the standard results.

To demonstrate the applicability of this theory we shall work out explicitly the simplest nontrivial example of higher-dimensional bosonization: a system of N noninteracting two-dimensional fermions in a harmonic oscillator potential. The single-particle hamiltonian is

$$H_{sp} = \frac{1}{2}(p_1^2 + x_1^2 + p_2^2 + x_2^2) \quad (22)$$

where for simplicity we chose the oscillator to be isotropic and of unit frequency. The single-body spectrum is the direct sum of two simple harmonic oscillator spectra, $E_{mn} = \hbar(m + n + 1)$, $m, n = 0, 1, \dots$. Calling $m + n + 1 = K$, the energy levels are $E_K = \hbar K$ with degeneracy K .

The N -body ground state consists of fermions filling states E_K up to a Fermi level $E_F = \hbar K_F$. In general, this state is degenerate, since the last energy level of degeneracy K_F is not fully occupied. Specifically, for a number of fermions N satisfying

$$N = \frac{K_F(K_F - 1)}{2} + M, \quad 0 \leq M \leq K_F \quad (23)$$

the Fermi sea consists of a fully filled bulk (the first term above) and M fermions on the K_F -degenerate level at the surface. The degeneracy of this many-body state is

$$g(K_F, M) = \frac{K_F!}{M!(K_F - M)!} \quad (24)$$

representing the ways to distribute the M last fermions over K_F states, and its energy is

$$E(K_F, M) = \hbar \frac{K_F(K_F - 1)(2K_F - 1)}{6} + \hbar K_F M \quad (25)$$

Clearly the vacua $(K_F, M = K_F)$ and $(K_F + 1, M = 0)$ are identical. Excitations over the Fermi sea come with energies in integer multiples of \hbar and degeneracies according to the possible fermion arrangements.

For the bosonized system we choose polar phase space variables,

$$h_i = \frac{1}{2}(p_i^2 + x_i^2), \quad \theta_i = \arctan \frac{x_i}{p_i}, \quad i = 1, 2 \quad (26)$$

in terms of which the single-particle hamiltonian and Poisson structure is

$$\{\theta_i, h_j\}_{sp} = \delta_{ij}, \quad H_{sp} = h_1 + h_2 \quad (27)$$

For the droplet description we take $h_1 = R$ and $\theta_1 = \sigma$ which leaves $\{h_2, \theta_2\} \sim \{x_2, p_2\}$ as the residual phase space. The bosonic hamiltonian is

$$H = \frac{1}{(2\pi\hbar)^2} \int d\sigma dh_2 d\theta_2 \left(\frac{1}{2} R^2 + h_2 R \right) \quad (28)$$

The ground state is a configuration with $R + h_2 = E_F$. The nonperturbative constraints $R > 0$, $h_2 > 0$ mean that the range of h_2 is $0 < h_2 < E_F$.

We ‘quantize’ the single-particle residual phase space (h_2, θ_2) by defining oscillator states $|a\rangle$, $a = 0, 1, 2, \dots$ satisfying $\hat{h}_2|a\rangle = \hbar(a + \frac{1}{2})|a\rangle$. The nonperturbative constraint for h_2 is implemented by restricting to the K_F -dimensional Hilbert space spanned by $a = 0, 1, \dots, K_F$ with $E_F = \hbar K_F - 1$. In the matrix representation R^{ab} becomes a $U(K_F)$ current algebra. We also Fourier transform in σ as in (18) into discrete modes R_n^{ab} , $n = 0, \pm 1, \dots$ (σ has a period 2π). The hamiltonian (28) has no matrix ordering ambiguities (being quadratic in R and h_2) but it needs quantum ordering. Just as in the $1+1$ -dimensional case, we normal order by pulling negative modes N to the left of positive ones. The result is

$$\frac{H}{\hbar} = \sum_{n < 0} R_{-n}^{ab} R_n^{ba} + \frac{1}{2} R_0^{ab} R_0^{ba} + (a + \frac{1}{2}) R_0^{aa} \quad (29)$$

To analyze the spectrum of (29) we perform the change of variables

$$\tilde{R}_n^{ab} = R_{n-a+b}^{ab} + (a - K_F + 1) \delta^{ab} \delta_n \quad (30)$$

The new fields \tilde{R} satisfy the same Kac-Moody algebra as R . The hamiltonian (29) becomes

$$\begin{aligned} \frac{H}{\hbar} = & \sum_{n < 0} \tilde{R}_{-n}^{ab} \tilde{R}_n^{ba} + \frac{1}{2} \tilde{R}_0^{ab} \tilde{R}_0^{ba} + (K_F - \frac{1}{2}) \tilde{R}_0^{aa} \\ & + \frac{K_F(K_F - 1)(2K_F - 1)}{6} \end{aligned} \quad (31)$$

The above is the standard quadratic form in \tilde{R} plus a constant and a term proportional to the $U(1)$ charge $\tilde{R}_0^{aa} = N - K_F(K_F - 1)/2$.

The ground state consists of the vacuum multiplet $|K_F, M\rangle$, annihilated by all positive modes \tilde{R}_n and transforming in the M -fold fully antisymmetric irrep of $SU(K_F)$ ($0 \leq M \leq K_F - 1$), with degeneracy equal to the dimension of this representation $K_F! / M!(K_F - M)!$. The $U(1)$ charge of \tilde{R} is given by the number of boxes in the Young tableau of the irreps, so it is M . The fermion number is, then, $N = K_F(K_F - 1)/2 + M$. Overall, we have a full correspondence with the many-body fermion ground states found before; the state $M = K_F$ is absent, consistently with the fact that the corresponding many-body state is the state $M = 0$ for a shifted K_F .

The energy of the above states consists of a constant plus a dynamical contribution from the zero mode \tilde{R}_0 . The quadratic part contributes $\frac{1}{2}\hbar M$, while the linear part contributes $\hbar(K_F - \frac{1}{2})M$. Overall, the energy is $\hbar K_F(K_F - 1)(2K_F - 1)/6 + \hbar K_F M$, also in agreement with the many-body result.

Excited states are obtained by acting with creation operators \tilde{R}_{-n} on the vacuum, introducing n quanta of energy. Due to the presence of zero-norm states, the corresponding Fock representation truncates in just the right way to reproduce the states of second-quantized fermions with an $SU(K_F)$ internal symmetry and fixed total fermion number. These are, again, into one-to-one correspondence with the excitation states of the many-body system, built as towers of one-dimensional excited Fermi seas over single-particle states $E_{m,n}$, one tower for each value of n , with the correct excitation energy. We have the nonperturbative constraint $0 \leq n < K_F$, as well as constraints related to the non-depletion of the Fermi sea for each value of n , just as in the one-dimensional case. The off-diagonal operators \tilde{R}_n^{ab} create particle transitions between towers, with the total particle number fixed to N by the value of the \tilde{R}_0 .

We conclude by stating that the proposed bosonized theory is defined in a hamiltonian setting. The transition to a lagrangian requires ‘inverting’ the canonical Poisson brackets to a two-form ω and corresponding connection $\omega = dA$. The resulting action will contain the noncommutative generalization of the Wess-Zumino term, ex-

pressed in terms of a star-unitary field U given by the star-exponential of a real field $U = \exp_*(i\Phi)$ [9]. This will be presented in a future publication.

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